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On the Norm of the Metric Projections

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Let X be a Banach space. Given M a subspace of X we denote with P_M the metric projection onto M. We define $\pi(X) := \sup \{ \|P_M\| : M$ a proximinal subspace of $X\}$. In this paper we give a bound for $\pi(X)$. In particular, when $X = L_p$, we obtain the inequality $\|P_M\| \leqslant 2^{|2/p-1|}$, for every subspace M of L_p. We also show that $\pi(X) = \pi(X^*)$. © 1999 Academic Press

1. INTRODUCTION AND NOTATIONS

Let X be a Banach space and M a subspace of X. We define

$$P_M(x) := \{ y \in M : \|x - y\| = d(x, M) \}.$$

The set-valued mapping $P_M : X \to 2^M$ thus defined is called the *metric projection* onto M. If $P_M(x) \neq \emptyset$ for every $x \in X$, we shall say that M is a *proximinal* subspace of X. We define the norm of P_M by

$$\|P_M\| := \sup\{ \|y\| : y \in P_M(x) \text{ and } \|x\| \leqslant 1 \}$$

and the *metric constant* $\pi(X)$ by

$$\pi(X) := \sup\{ \|P_M\| : M \text{ a proximinal subspace of } X \}.$$

We trivially have $1 \leqslant \pi(X) \leqslant 2$. It is well known that if $\dim X \geqslant 3$ then X is a Hilbert space iff $\pi(X) = 1$, see [3, Theorem 5.1].

A *uniformly non-square* space is a Banach space X such that there exists a constant α, $0 < \alpha < 1$, satisfying $\|x - y\| \leqslant 2\alpha$ or $\|x + y\| \leqslant 2\alpha$ for every

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$x, y \in B_X$, where B_X is the unit ball. In [2, Theorem 1.1] Cuenya and Mazzone proved that X is an uniformly non-square space iff $\pi(X) < 2$. So, we observe that the metric constant $\pi(X)$ is close to the geometry of the Banach space X.

The term *operator* means a bounded linear operator. Two Banach spaces X and Y are called *isomorphic* if there exists an invertible operator from X onto Y. The *Banach–Mazur distance* $d(X, Y)$ is defined by $\inf \|T\| \, \|T^{-1}\|$, the infimum being taken over all invertible operators from X onto Y (if X is not isomorphic to Y we put $d(X, Y) = \infty$).

As is usual, for $1 \leqslant p \leqslant \infty$ and $n \in \mathbb{N}$, we let l_p^n denote the linear space $\mathbb{R}^n$ with the l_p-norm. We have the following theorems.

THEOREM 1.1. (a) *Let X be an n-dimensional subspace of $L_p(\mu)$, $1 \leqslant p \leqslant \infty$. Then $d(X, l_2^n) \leqslant n^{|1/2 - 1/p|}$.*

(b) *For any n-dimensional space X, $d(X, l_2^n) \leqslant \sqrt{n}$.*

Proof. See [4, Corollary III.B.9].

We define the constant $\mu(X)$ by

$$\mu(X) = \sup\{d(E, l_2^2) \colon E \subset X \text{ and } \dim \ E = 2\}.$$

We mention now the two main points of this article. In Section 2 of this paper we prove with Theorem 2.2 that $\pi(X) \leqslant (\mu(X))^2$. In Section 3 we show the relation $\pi(X) = \pi(X^*)$.

2. A BOUND FOR $\pi(X)$

We need to prove the following lemma.

LEMMA 2.1. *Let $\mathscr{H}$ be a Hilbert space and M a proximinal subspace of $\mathscr{H}$. Let $x \in \mathscr{H}$ and $1 \leqslant k < \infty$. We suppose that $y \in M$ satisfies*

$$\|x - y\| \leqslant k d(x, M).$$

Then

$$\|y\| \leqslant k \, \|x\|.$$

Proof. We can assume that $x \neq 0$. Let $z = P_M(x)$. We put $d := d(x, M) = \|x - z\|$. As $x - z \perp y$ we obtain

$$\begin{aligned}\|y\| &\leqslant \|z\| + \|y-z\| = \sqrt{\|x\|^2 - d^2} + \sqrt{\|y-x\|^2 - d^2} \\ &\leqslant \sqrt{\|x\|^2 - d^2} + \sqrt{k^2 d^2 - d^2} \\ &= \|x\| \left(\sqrt{1 - \left(\frac{d}{\|x\|} \right)^2} + \frac{d}{\|x\|} \sqrt{k^2 - 1} \right). \end{aligned} \tag{1}$$

It is easy to prove that the function $f(\xi) := \sqrt{1-\xi^2} + \xi \sqrt{k^2-1}$ for $\xi \in [0, 1]$ has a maximum at $\xi_0 = \sqrt{1-(1/k^2)}$ and $f(\xi_0) = k$. By (1), this implies the desired result. ▮

THEOREM 2.2. *For every Banach space X,*

$$\pi(X) \leqslant (\mu(X))^2. \tag{2}$$

Proof. Let M be a proximinal subspace of X, $x \in X \setminus M$, and $y \in P_M(x)$. Let E be the subspace spanned by x and y and $T: E \to l_2^2$ be an invertible operator. For all $\alpha \in \mathbb{R}$, we have $\|x-y\| \leqslant \|x - \alpha y\|$. As

$$\frac{1}{\|T^{-1}\|} \|z\| \leqslant \|T(z)\| \leqslant \|T\| \, \|z\| \qquad \text{for all} \quad z \in E. \tag{3}$$

We get, for all $\alpha \in \mathbb{R}$,

$$\|T(x) - T(y)\| \leqslant \|T\| \, \|x-y\| \leqslant \|T\| \, \|x - \alpha y\| \leqslant \|T\| \, \|T^{-1}\| \, \|T(x) - \alpha T(y)\|.$$

Using Lemma 2.1, for $\mathscr{H} = l_2^2$ and M the subspace generated by $T(y)$, we infer that

$$\|T(y)\| \leqslant \|T\| \, \|T^{-1}\| \, \|T(x)\|.$$

By (3) we get

$$\|y\| \leqslant (\|T\| \, \|T^{-1}\|)^2 \, \|x\|.$$

Taking the infimum over all T we obtain

$$\|y\| \leqslant (d(E, l_2^2))^2 \, \|x\|.$$

This proves the theorem. ▮

The next result follows immediately from Theorems 1.1 and 2.2.

COROLLARY 2.3. $\pi(L_p) \leqslant 2^{|2/p - 1|}$.

We trivially have $\pi(L_p)=2^{|2/p-1|}$ for $p=1, 2, \infty$. We will show that $\pi(l_p^2)<2^{|2/p-1|}$ for others values of p. Let $1<p<\infty$, $p\neq 2$, and $q=p/(p-1)$. By Theorem 3.4 below we can suppose $p<2<q$. Let H be the hyperplane in $\mathbb{R}^2$ given by $\{(x, y): \alpha x+\beta y=0\}$, where $\|(\alpha, \beta)\|_q=1$. If $\alpha=0$ or $\beta=0$ then $\|P_H\|=1$. We suppose $\alpha, \beta\neq 0$. From [2, Lemma 2.1] we have

$$\|P_H\|=(|\alpha|^{(q-1)q}+|\beta|^{(q-1)q})^{1/q}\,\|(\alpha, \beta)\|_p<\|(\alpha, \beta)\|_p\leqslant 2^{|2/p-1|}.$$

Therefore

$$\pi(l_p^2)=\max\{\|P_H\|: H \text{ hyperplane of } \mathbb{R}^2\}<2^{|2/p-1|}.$$

3. $\pi(X)=\pi(X^*)$

Let X and Y two Banach spaces. We say that the set-valued mapping $P: X\to 2^Y$ admits a *linear selection* if there exists a linear mapping $T: X\to Y$ such that $T(x)\in P(x)$, for every $x\in X$. We observe that, if the metric projection $P_M: X\to M$ admits a linear selection T, then T is a bounded operator. In fact, $\|T\|\leqslant\pi(X)$. As is usual, by T^* we denote the adjoint operator of T. We recall that if M is a subspace of X, we denote by $M^\perp$ the subspace of X^* defined by $M^\perp:=\{x^*\in X^*: x^*(x)=0 \text{ for all } x\in M\}$.

LEMMA 3.1. *Let M be a subspace of a Banach space X. We suppose that the metric projection P_M admits a linear selection T. Then T^* is a linear selection of $P_{N^\perp}$, where $N:=\ker T$.*

Proof. As T is an operator with $T^2=T$ (i.e., is a projection) we have $X=M\oplus N$ and N is a closed subspace of X. Moreover, it is easy to see that $T^*(X^*)=N^\perp$. Now we will show that, for any $x^*\in X^*$, $T^*(x^*)\in P_{N^\perp}(x^*)$. For every $x^*\in X^*$ we have

$$\begin{aligned}\|x^*-T^*(x^*)\|&=\sup\{|(x^*-T^*(x^*))(x+y)|: \|x+y\|\leqslant 1, x\in M \text{ and } y\in N\}\\&=\sup\{|x^*(y)|: \|x+y\|\leqslant 1, x\in M \text{ and } y\in N\}.\end{aligned}\tag{4}$$

If $y\in N$ then $0=T(y)\in P_M(y)$. Hence $\|y\|\leqslant\|x+y\|$, for all $x\in M$. Then

$$\begin{aligned}&\sup\{|x^*(y)|: \|x+y\|\leqslant 1, x\in M \text{ and } y\in N\}\\&\quad=\sup\{|x^*(y)|: \|y\|\leqslant 1 \text{ and } y\in N\}.\end{aligned}\tag{5}$$

Let $z^* \in N^{\perp}$. Using (4) and (5) we obtain

$$\begin{aligned}\|x^* - z^*\| &\geqslant \sup\{|(x^* - z^*)(y)|: \|y\| \leqslant 1 \text{ and } y \in N\}\\ &= \sup\{|x^*(y)|: \|y\| \leqslant 1 \text{ and } y \in N\}\\ &= \|x^* - T^*(x^*)\|.\end{aligned}$$

Consequently $T^*(x^*) \in P_{N^{\perp}}(x^*)$. ▮

In [3, p. 142] we see, implicitly, the following lemma.

LEMMA 3.2. *Let M be a proximinal hyperplane of the Banach space X, $x \in X$, and $y \in P_M(x)$. Then P_M admits a linear selection T with $T(x) = y$.*

Now we will present the main result of this section.

THEOREM 3.3. *For every Banach space X we have that $\pi(X) = \pi(X^*)$.*

Proof. We can suppose that X is a reflexive Banach space. For if X is non-reflexive then X is not a uniformly non-square Banach space, [1, p. 256]. Therefore $\pi(X) = 2$. We observe that X^* is a non-reflexive space. Then, in similar way, $\pi(X^*) = 2$. Thus $\pi(X) = \pi(X^*)$. We recall that if X is reflexive then every closed subspace of X is proximinal, see [3, p. 99].

For every $\varepsilon > 0$ we can find a subspace M of X, $x \in B_X$, and $y \in P_M(x)$ such that $\pi(X) - \varepsilon < \|y\|$. From the characterization theorem of elements of best approximation, [3, p. 18], we get $x^* \in M^{\perp}$ with $\|x^*\| = 1$ and $x^*(x - y) = \|x - y\|$. We put $H = \operatorname{Ker} x^*$. Then $y \in P_H(x)$. As H is a proximinal subspace of X, as a consequence of Lemma 3.2 we get a linear selection T of P_H such that $T(x) = y$. From Lemma 3.1 we obtain

$$\pi(X) - \varepsilon \leqslant \|y\| = \|T(x)\| \leqslant \|T\| = \|T^*\| \leqslant \|P_{N^{\perp}}\| \leqslant \pi(X^*)$$

where $N := \operatorname{Ker} T$. Since ε is arbitrary we have

$$\pi(X) \leqslant \pi(X^*). \tag{6}$$

From inequality (6) we infer that $\pi(X^*) \leqslant \pi(X^{**})$. Moreover, as X is a reflexive Banach space, we have that $\pi(X^{**}) = \pi(X)$. Consequently $\pi(X) = \pi(X^*)$. ▮

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